

HIGHER HICKERSON FORMULA

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ABSTRACT. In [11], Hickerson made an explicit formula for Dedekind sums $s(p, q)$ in terms of the continued fraction of p/q . We develop analogous formula for generalized Dedekind sums $s_{i,j}(p, q)$ defined in association with the $x^i y^j$ -coefficient of the Todd power series of the lattice cone in \mathbb{R}^2 generated by $(1, 0)$ and (p, q) . The formula generalizes Hickerson's original one and reduces to Hickerson's for $i = j = 1$. In the formula, generalized Dedekind sums are divided into two parts: the integral $s_{ij}^I(p, q)$ and the fractional $s_{ij}^R(p, q)$. We apply the formula to Siegel's formula for partial zeta values at a negative integer and obtain a new expression which involves only $s_{ij}^I(p, q)$ the integral part of generalized Dedekind sums. This formula directly generalize Meyer's formula for the special value at 0. Using our formula, we present the table of the partial zeta value at $s = -1$ and -2 in more explicit form. Finally, we present another application on the equidistribution property of the fractional parts of the graph $\left(\frac{p}{q}, R_{i+j} q^{i+j-2} s_{ij}(p, q)\right)$ for a certain integer R_{i+j} depending on $i + j$.

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1. INTRODUCTION

In [11], Hickerson obtained an explicit formula for Dedekind sum $s(p, q)$ in terms of the elements of the continued fraction of p/q , where the Dedekind sum $s(p, q)$ is defined as

$$s(p, q) := \sum_{k=0}^{q-1} \left(\left(\frac{k}{q} \right) \right) \left(\left(\frac{pk}{q} \right) \right).$$

Date: June 7, 2016.

J.L. was supported by the Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(NRF-2011-0023688) and (NRF-2009-0093827).

B.J. was supported by NRF-2015R1D1A1A09059083.

H.C. was supported by 2014 Hongik University Research Fund.

Here $((-))$ denotes the sawtooth function (ie. for $x \in \mathbb{R}$, $((x)) = x - [x] - \frac{1}{2}$ if $x \notin \mathbb{Z}$ and $((x)) = 0$ otherwise). Hickerson's formula is written as follows:

$$(1) \quad 12s(p, q) = \begin{cases} \frac{p-q_{n-1}}{q} + \sum_{i=1}^n (-1)^{i+1} a_i & \text{if } n \text{ is even,} \\ \frac{p+q_{n-1}}{q} + \sum_{i=1}^n (-1)^{i+1} a_i - 3 & \text{if } n \text{ is odd.} \end{cases}$$

where $p_k/q_k = [a_0, \dots, a_k]$ be the k -th convergent of p/q and a_i are the terms of the continued fraction of p/q :

$$(2) \quad \frac{p}{q} = [a_0, \dots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{\dots + \frac{1}{a_n}}}$$

The idea of the proof is successive application of the celebrated reciprocity formula of Dedekind sums:

$$s(p, q) + s(q, p) = \frac{1}{12}(p/q + q/p + 1/pq) - 1/4$$

Hickerson used the formula to show the density of the graph of Dedekind sums $\{(p/q, s(p, q)) \in \mathbb{R}^2 | (p, q) = 1, q > 0\}$ in *loc.cit.*. It implies a conjecture of Rademacher saying that $s(p, q)$ is dense in \mathbb{R} ([19]).

In Hickerson's formula (1), the Dedekind sum is divided into integral and fractional parts. Both are interesting in opposite directions. Integral part appears in the formula for ζ or L -values of real quadratic fields. Namely, Meyer's formula for partial zeta value of an ideal \mathfrak{a} of a real quadratic field at $s = 0$ is as follows

$$\zeta(\mathfrak{a}, 0) = \begin{cases} \frac{1}{12} \sum_{i=1}^n (-1)^{i+1} a_i & \text{if } n \text{ is even,} \\ \frac{1}{12} \sum_{i=1}^n (-1)^{i+1} a_i - 3 & \text{if } n \text{ is odd.} \end{cases}$$

where a_i are terms of the continued fraction of a rational number p/q in relation with \mathfrak{a} (cf. [16], [22], [13], [15]). We are not going to give exact description of the numbers appearing here as well as the definition of the partial zeta function but refer to Sec.6 of this article. The special value is exactly the integer part of $12s(p, q)$ as appeared in (1) but a universal denominator 12 independent of \mathfrak{a} . The occurrence of 12 is explained in [14]. The fractional part (mod 1) is given as a Laurent polynomial in p divided by q and is necessary in relating the Kloosterman sum to Dedekind sums:

$$\sum_{\substack{0 < p < q \\ \gcd(p, q) = 1}} \exp(2\pi i (12s(p, q))) = \sum_{p \in (\mathbb{Z}/q\mathbb{Z})^*} \exp\left(2\pi i \frac{p + p^{-1}}{q}\right)$$

Then the Weil bound for Kloosterman sum implies the Weyl equidistribution of the fractional part of $12s(p, q)$ in the unit interval.

For partial zeta values at strictly negative integers, there is another formula due to Siegel ([20], see also [6]), but one needs generalized Dedekind sums:

$$(3) \quad s_{ij}(p, q) := \sum_{k=0}^{q-1} \bar{B}_i\left(\frac{k}{q}\right) \bar{B}_j\left(\frac{pk}{q}\right).$$

Here $\bar{B}_i(-)$ is the i -th periodic Bernoulli function. These sums $s_{ij}(p, q)$ vanish for odd $N = i + j$. For the Siegel formula, we refer the reader to Sec.6. of this article. The case one of $j = 1$ is known as Apostol sums. These sums are studied by Apostol and Carlitz and appear in modular transformation of generalized Lambert series (cf. [1], [2], [4], [5]).

The goal of this paper is to obtain a higher degree analogue of the Hickerson's formula. It is an explicit formula for the generalized Dedekind sums using the terms of the continued fraction of p/q

as in Hickerson's. Furthermore, the formula should feature separation between the integral and the fractional part in the following way:

$$(4) \quad s_{i,j}(p, q) = \frac{1}{q^{i+j-2}} s_{i,j}^I(p, q) + \frac{1}{q^{i+j-1}} s_{i,j}^R(p, q).$$

Here, $s_{i,j}^I$ and $s_{i,j}^R$ denote the integral and the fractional part which are written in terms of the terms of the continued fraction of p/q and belong to $\frac{1}{R_{i+j}}\mathbb{Z}$ where R_{i+j} is an integer fixed by $N = i + j$ (R_{i+j} is given explicitly in (7)). Bearing in mind the two results aforementioned, which are obtained from the formulae of the integral and the fractional parts, we are going to present two applications in each direction.

The main result of [12] shows that $R_{i+j}s_{i,j}^R(p, q)/q \pmod{1}$ (thus same as $R_{i+1}q^{i+j-2}s_{i,j}(p, q) \pmod{1}$) is equidistributed in the unit interval. *In loc. cit.*, explicit formula for $s_{i,j}^R(p, q)$ is given. In this paper, it remains to find explicit formula of $s_{i,j}^I(p, q)$, that will complement the previous result. For $i = j = 1$, it is reduced to the case of the classical Dedekind sums as appeared in Hickerson([11]) with $R_2 = R_{i+j} = 12$. It is crucial ingredient to single out the integral part from the classical Dedekind sums in the proof of Density of the graph of Dedekind sums in \mathbb{R}^2 . Analogous formula for the generalized Dedekind sums can be used in showing the equidistribution of the graph $\left(\frac{p}{q}, R_{i+j}q^{i+j-2}s_{i,j}(p, q)\right) \pmod{\mathbb{Z} \times \mathbb{Z}}$. It will be an application of the main theorem and appear in Sec. 7. For classical Dedekind sums, one has much stronger equidistribution for the fractional part by results of Myerson and Vardi(cf. [17], [23]). One can take R_{i+j} as arbitrary nonzero real.

When $i, j \neq 1$, it is not so simple as in Hickerson's due to lack of nice reciprocity formula relating $s_{i,j}(p, q)$ and $s_{i,j}(q, p)$. If either i or j is 1, the generalized Dedekind sums are studied by Apostol and Carlitz. There is a reciprocity formula found by Apostol (cf. [1], [2], [4], [5]). In general as reciprocity, the best is the one involving other sums with same $N = i + j$ at the same time. We notice that the generalized Dedekind sums $s_{i,j}(p, q)$ appear to be closely related to the coefficients of the Todd series $\text{Todd}_{pq}(x_1, x_2)$ of a lattice cone generated by $e_1 = (1, 0)$ and (p, q) (For the definition of Todd series, see Sec. 2). Additionally, the association of (normalized) Todd series S_σ to a lattice cone σ has additivity.

$$(5) \quad \begin{aligned} \Phi : \sigma &\mapsto S_\sigma(A_\sigma^{-1}(x, y)^t), \\ \Phi(\sigma(v_1, v_2)) + \Phi(\sigma(v_2, v_3)) &= \Phi(\sigma(v_1, v_3)). \end{aligned}$$

Here, for two primitive lattice vector v, w , $\sigma(v, w)$ denotes the lattice cone generated by v, w . In this context, the reciprocity formulae are all consequences of this additivity. The first quadrant cone of \mathbb{R}^2 can be divided into two by putting a vector (p, q) for p, q relative prime and both positive. Then one can relate $\text{Todd}_{pq}(x, y)$ and $\text{Todd}_{qp}(x, y)$. Reading the coefficient of particular monomial, one can recover the reciprocity formula including other generalized Dedekind sums of the same weight(cf. [10], [21]).

The strategy of this paper is to lift the reciprocity to the level of the Todd series. As is well-known, the continued fraction of p/q describes a decomposition of the cone given by (p, q) into nonsingular cones. At the final step, we obtain the explicit formula for generalized Dedekind sums in terms of the continued fraction of p/q .

The first of our main result is the following explicit formula:

Theorem 1.1. *Suppose e, f is a positive integer and $N = e + f$ is even. Then we have*

$$s_{e,f}(p, q) + \delta(e, f)B_e B_f = \frac{1}{q^{N-2}} s_{e,f}^I(p, q) + \frac{1}{q^{N-1}} s_{e,f}^R(p, q),$$

where $s_{e,f}^I(p, q)$ and $s_{e,f}^R(p, q)$ are given as:

$$s_{e,f}^I(p, q) = e!f! \sum_{\alpha+\beta=e} \left[\sum_{k=-1}^{n-1} (-1)^{k+1} f_k(\alpha, \beta) + \frac{B_N}{N!} \sum_{k=0}^{n-1} (-1)^k a_{k+1} g_k(\alpha, \beta) \right],$$

$$s_{e,f}^R(p, q) = e!f! \frac{B_N}{N!} \left[\binom{N-1}{f} p^f + (-1)^{e(n-1)} \binom{N-1}{e} (q_{n-1})^e \right]$$

Here

$$D_k := pq_k - qp_k, \quad \text{for } k = 0, 1, \dots, n-1$$

and

$$f_k(\alpha, \beta) := \frac{q_k^\beta q_{k+1}^\alpha}{\alpha! \beta!} \sum_{i=\alpha}^{N-2-\beta} \frac{(-1)^i}{(N-2-i-\beta)!(i-\alpha)!(i+1)} \frac{B_{i+1}}{(N-i-1)} \frac{B_{N-i-1}}{(N-i-1)} D_k^{N-2-i-\beta} D_{k+1}^{i-\alpha}$$

$$g_k(\alpha, \beta) := \frac{q_{k-1}^\beta q_{k+1}^\alpha}{\alpha! \beta!} \sum_{i=\alpha}^{N-2-\beta} \frac{(N-i-2)!}{(N-2-i-\beta)!} \frac{i!}{(i-\alpha)!} D_{k-1}^{N-i-2-\beta} D_{k+1}^{i-\alpha}$$

and $\delta(i, j) = \begin{cases} 1, & i = 1 \text{ or } j = 1, \\ 0, & \text{otherwise} \end{cases}$ and B_i is the i -th Bernoulli number.

Using this formula, we present here two applications. First, we apply the explicit formula to the Siegel formula for the partial zeta value of an ideal of a real quadratic field at nonpositive integer and obtain a modified one. Surprisingly, $s_{i,j}^R(p, q)$ in the formula cancels each other and the new formula does not involve $s_{i,j}^R(p, q)$ the fractional part of $s_{i,j}(p, q)$ at all. As seen in Thm.1.1, it appears as a polynomial in terms of the continued fraction of p/q same as in Meyer's formula. Actually, restricting $i = j = 1$, one can recover the formula of Meyer.

Theorem 1.2 (Higher Meyer formula). *Let $\mathfrak{a} = [\alpha, \beta]$ be an ideal of a real quadratic field and ϵ be the totally positive fundamental unit of K . If the multiplication of ϵ^{-1} is identified with a matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ w.r.t. basis $\{\alpha, \beta\}$, then we have*

$$(6) \quad \zeta(\mathfrak{a}, 1-N) = \frac{(-1)^N}{q^{2N-2}} \sum_{k=0}^{2N-2} (-1)^{k+1} e_{N,k} \frac{s_{k+1, 2N-k-1}^I(p, q)}{(k+1)(2N-1-k)}$$

where $e_{N,k}$ is given by $(z^2 + (p+s)z + 1)^{N-1} = \sum_{i=0}^{2N-2} e_{N,k} z^k$.

The second application is the distribution of the graph of the fractional part $R_{i+j} q^{i+j-2} s_{i,j}(p, q) \pmod{1}$ as function of p/q . Using Thm.1.1., $R_{i+j} q^{i+j-2} s_{i,j}(p, q) \pmod{1}$ equals $R_{i+j} q^{i+j-2} s_{i,j}^R(p, q)$.

Theorem 1.3 (Equidistribution). *The fractional parts of a graph $\left(\frac{p}{q}, R_{i+j} q^{i+j-2} s_{i,j}(p, q)\right)$ are equidistributed in $[0, 1)^2$, where*

$$(7) \quad R_{i+j} := \binom{N}{i} \beta_N r_N,$$

and $\frac{\alpha_k}{\beta_k}$ is the reduced fraction of $B_k \neq 0$ with $\beta_k > 0$ and

$$r_N := \text{L.C.M.} \left\{ \text{Denominator of } \beta_N \binom{N}{i+1} B_{i+1} B_{N-i-1} \mid i \text{ odd}, 0 \leq i \leq N-2 \right\}.$$

For higher dimension, we have an analogue of not only the Dedekind sums but also the generalized Dedekind sums ([25], [9], [7]). In [7], the decomposition of higher dimensional Dedekind sums into integral and rational parts is given but only the rational part is explicit.

This paper is composed as follows. In Sec. 2, we briefly recall the notion of Todd series and its cocycle property. We relate the generalized Dedekind sums to the coefficients of Todd series. In Sec. 3, we relate a cone decomposition to a continued fraction. In Sec.4, we use the results of earlier sections to obtain an explicit formula lifting the Hickerson's formula to the level of Todd series. In Sec. 5, we give the proof of Thm.1.1 by descending to the level of generalized Dedekind sums. Sec.6 is devoted to the proof of Thm.1.2 and contains a table for partial zeta values in our formula. Finally in Sec.7, we prove Thm.1.3.

ACKNOWLEDGEMENTS

We are grateful to Vincent Maillot and Haesang Sun for various discussions and encouragement.

2. TODD SERIES AND GENERALIZED DEDEKIND SUMS

Let M be the standard lattice \mathbb{Z}^2 in \mathbb{R}^2 . Let $\sigma = \sigma(v_1, v_2)$ be a lattice cone with primitive lattices v_1, v_2 bounding σ . We take the orientation (ie. the order of the rays) into consideration so that

$$\sigma(v_1, v_2) = -\sigma(v_2, v_1).$$

Let A_σ be an integer coefficient matrix whose columns are the lattice vectors v_1, v_2 in \mathbb{Z}^2 . We will often identify with a lattice cone σ with its matrix A_σ . M_σ denotes the sublattice of M generated by v_1, v_2 . $\Gamma_\sigma := M/M_\sigma$ is isomorphic to a cyclic group of order $|\det(A_\sigma)|$.

For $g \in M$ representing $\gamma \in \Gamma_\sigma$, we have rational numbers $a_{\sigma,i}(g)$, $i = 1, 2$ such that

$$g = a_{\sigma,1}(\gamma)v_1 + a_{\sigma,2}(\gamma)v_2.$$

Since $a_{i,\sigma}$ is integral on M_σ , we have a character $\chi_{\sigma,i}$ on Γ_σ as

$$\chi_{\sigma,i}(\gamma) := e(a_{\sigma,i}(g)), \quad \text{for } i = 1, 2.$$

Definition 2.1. *The Todd power series of σ is defined as:*

$$(8) \quad \text{Todd}_\sigma(x_1, x_2) := \sum_{\gamma \in \Gamma_\sigma} \frac{x_1}{1 - \chi_{\sigma,1}(\gamma)e^{-x_1}} \frac{x_2}{1 - \chi_{\sigma,2}(\gamma)e^{-x_2}}.$$

The Todd series is analytic in a neighborhood of $0 \in \mathbb{C}^2$ and has Taylor expansion. Note that the Todd series is invariant of the $\text{GL}_2(\mathbb{Z})$ equivalent class of cones by the following proposition (ie. for $A \in \text{GL}_2(\mathbb{Z})$, $\text{Todd}_\sigma(x_1, x_2) = \text{Todd}_{A\sigma}(x_1, x_2)$).

If $p, q > 0$ are a pair of nonnegative integers prime to each other, $(1, 0)$ and (p, q) are primitive lattice vectors and linearly independent. Let σ_{pq} denote the cone generated by $(1, 0)$ and (p, q) . We shall write Todd_{pq} instead of $\text{Todd}_{\sigma_{pq}}$ for simplicity.

Then the generalized Dedekind sum $s_{ij}(p, q)$ is identified with the $x_1^i x_2^j$ -coefficient of Todd_{pq} . For $i = j = 1$, it is the classical Dedekind sum: $s_{11}(p, q) = s(p, q)$. Let us denote the $x_1^i x_2^j$ -coefficient of $\text{Todd}_{pq}(x_1, x_2)$ by $t_{ij}(p, q)$.

$$(9) \quad \text{Todd}_{pq}(x_1, x_2) = \sum_{i,j \geq 0} \frac{t_{ij}(p, q)}{i!j!} x_1^i x_2^j.$$

Then we have

$$(10) \quad t_{ij}(p, q) = -(-q)^{i+j-1} (s_{ij}(p, q) + \delta(i, j)B_i B_j),$$

where $\delta(i, j) = \begin{cases} 1, & i = 1 \text{ or } j = 1, \\ 0, & \text{otherwise} \end{cases}$ and B_i is the i -th Bernoulli number.

For a lattice cone σ , its normalized Todd series is defined as follows:

Definition 2.2. The normalized Todd series $S_\sigma(x_1, x_2)$ of σ is defined as

$$S_\sigma(x_1, x_2) = \frac{1}{\det(A_\sigma)x_1x_2} \text{Todd}_\sigma(x_1, x_2).$$

Similarly, $S_{\sigma_{pq}}$ is abbreviated to S_{pq} as in unnormalized case.

$S_\sigma(x_1, x_2)$ is a Laurent series in variables x_1, x_2 and has simple poles along $x_1 = 0$ and $x_2 = 0$. Also note that swapping two rays of the cone interchanges not only the variables but also the sign in S_σ . Thus the orientation is reflected in S_σ and changes the sign(cf. [14]).

$$(11) \quad S_{-\sigma}(x_1, x_2) = -S_\sigma(x_2, x_1)$$

$$(12) \quad \text{Todd}_{-\sigma}(x_1, x_2) = \text{Todd}_\sigma(x_2, x_1)$$

Proposition 2.3 (Todd cocycle). If we set $\Phi(\sigma) = S_\sigma(A_\sigma^{-1}(x_1, x_2))$, then we have

$$\Phi(\sigma_1 + \sigma_2) = \Phi(\sigma_1) + \Phi(\sigma_2).$$

Proof. For the proof, we refer the reader to Thm.3 of [18]. □

3. CONTINUED FRACTION AND CONE DECOMPOSITION

In this section, we would like to relate nonsingular decomposition of $\sigma_{p,q}$ with the continued fraction of p/q as is done in [3]. According to the cocycle property of the (normalized) Todd series, the (normalized) Todd series is decomposed into sum of those of nonsingular cones.

Suppose q and p be relatively prime positive integers and suppose $q > p$. Let p/q have the following continued fraction expansion:

$$\frac{q}{p} = a_1 + \frac{1}{a_2 + \cdots \frac{1}{a_n}}$$

where $a_i \geq 1$ are all positive integers. Put $(p_{-1}, q_{-1}) = (1, 0)$, $(p_0, q_0) = (0, 1)$. For $i \geq 1$, p_i/q_i be the i -th convergent of p/q :

$$\frac{q_i}{p_i} := a_1 + \frac{1}{a_2 + \cdots \frac{1}{a_i}}.$$

p_i and q_i are uniquely determined by assuming $q_i > 0$ and $\gcd(p_i, q_i) = 1$. (p_i, q_i) is computed by iterating elementary Möbius transformations $A_i = \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}$ in $\text{GL}_2(\mathbb{Z})$ in the following manner:

$$(13) \quad \begin{pmatrix} p_i \\ q_i \end{pmatrix} = A_1 A_2 \cdots A_i \begin{pmatrix} 1 \\ 0 \end{pmatrix} = A_1 A_2 \cdots A_{i-1} \begin{pmatrix} a_i \\ 1 \end{pmatrix}$$

Comparing (p_k, q_k) , (p_{k-1}, q_{k-1}) and (p_{k+1}, q_{k+1}) using above formula, we obtain

$$(14) \quad (p_{k+1}, q_{k+1}) = (p_{k-1}, q_{k-1}) + a_{k+1}(p_k, q_k)$$

which will be used in the next section.

Note that $v_k = (p_k, q_k)$ for $-1 \leq k \leq n$ is primitive lattice vector in the 1st quadrant. We have the following (virtual) cone decomposition of σ into nonsingular cones:

$$\sigma := \sigma_{pq} = \sigma(v_{-1}, v_n) = \sum_{k=-1}^{n-1} \sigma_k,$$

where $\sigma_k := \sigma(v_k, v_{k+1})$.

4. NONSINGULAR DECOMPOSITION OF TODD SERIES

From the additivity of normalized Todd series, we have

$$\Phi(\sigma_{p,q}) = \sum_{k=-1}^{n-2} \Phi(\sigma_k),$$

where

$$\Phi(\sigma) = S_\sigma(A_\sigma^{-1}(x, y)).$$

For $\sigma_e = \sigma((1, 0), (0, 1))$, from $\text{GL}_2(\mathbb{Z})$ -invariance of Todd power series, we have

$$\begin{aligned} S_{\sigma_k} &= S_{A_{\sigma_k} \sigma_e} = \frac{1}{\det A_{\sigma_k} xy} \text{Todd}_{A_{\sigma_k} \sigma_e}(x, y) \\ &= \frac{1}{\det A_{\sigma_k} xy} \text{Todd}_{\sigma_e}(x, y) = (-1)^{k+1} \frac{\text{Todd}(x, y)}{xy}. \end{aligned}$$

Thus we obtain the following expression:

$$(15) \quad S_{pq}(x, y) = \sum_{k=-1}^{n-1} (-1)^{k+1} F \left(A_{\sigma_k}^{-1} A_\sigma(x, y)^t \right),$$

where

$$F(x, y) = \frac{1}{1 - e^{-x}} \frac{1}{1 - e^{-y}} = \frac{\text{Todd}(x, y)}{xy}.$$

For $-1 \leq k \leq n-1$ let $M_k(x, y)$ be linear forms in x and y defined as follows:

$$(16) \quad A_{\sigma_k}^{-1} A_\sigma \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} M_{k+1}(x, y) \\ M_k(x, y) \end{pmatrix}$$

Since $A_{\sigma_k}^{-1} \in \text{GL}_2(\mathbb{Z})$, $M_k(x, y)$ has coefficients in \mathbb{Z} .

As $\det(A_\sigma) = q$, by multiplying qxy , we have

$$\begin{aligned} (17) \quad \text{Todd}_{pq}(x, y) &= qxy \sum_{k=-1}^{n-1} (-1)^{k+1} \frac{\text{Todd}(M_k, M_{k+1})}{M_k M_{k+1}} \\ &= qxy \sum_{k=-1}^{n-1} (-1)^{k+1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} \frac{B_i B_j}{i! j!} M_k^{j-1} M_{k+1}^{i-1}. \end{aligned}$$

Denote by Todd_σ^N the degree N homogeneous part of Todd_σ . Then from (17) Todd_σ^N is given as follows:

$$\begin{aligned} (18) \quad \text{Todd}_\sigma^N &= qxy \sum_{k=-1}^{n-1} (-1)^{k+1} \sum_{i=0}^{N-2} (-1)^N \frac{B_{i+1}}{(i+1)!} \frac{B_{N-i-1}}{(N-i-1)!} M_k^{N-2-i} M_{k+1}^i \\ &\quad + (-1)^N qxy \frac{B_N}{N!} \sum_{k=-1}^{n-1} (-1)^{k+1} \frac{M_k^N + M_{k+1}^N}{M_k M_{k+1}}. \end{aligned}$$

From (14), for $k \geq 0$ we have

$$M_{k-1} - M_{k+1} = a_{k+1}M_k.$$

and

$$(19) \quad \begin{aligned} & qxy \sum_{k=-1}^{n-1} (-1)^{k+1} \frac{M_k^N + M_{k+1}^N}{M_k M_{k+1}} \\ &= qxy \left(\sum_{k=0}^{n-1} (-1)^k a_{k+1} \sum_{i=0}^{N-2} M_{k-1}^{N-2-i} M_{k+1}^i \right) + M_0^{N-1}x + M_{n-1}^{N-1}y. \end{aligned}$$

Therefore, plugging (19) into (18), we obtain

$$(20) \quad \begin{aligned} \text{Todd}_{pq}^N &= qxy \sum_{k=-1}^{n-1} (-1)^{k+1} \sum_{i=0}^{N-2} (-1)^N \frac{B_{i+1}}{(i+1)!} \frac{B_{N-i-1}}{(N-i-1)!} M_k^{N-2-i} M_{k+1}^i \\ &+ (-1)^N q \frac{B_N}{N!} xy \left(\sum_{k=0}^{n-1} (-1)^k a_{k+1} \sum_{i=0}^{N-2} M_{k-1}^{N-i-2} M_{k+1}^i \right) + \frac{B_N}{N!} M_0^{N-1}x + \frac{B_N}{N!} M_{n-1}^{N-1}y. \end{aligned}$$

5. COMPUTATION OF DEDEKIND SUMS

In this section, using the result of earlier sections, we give a proof of Thm.1.1.

Let m_k and ℓ_k be the coefficients of M_k : $M_k(x, y) = m_k x + \ell_k y$. We already know that m_k and ℓ_k are determined by the continued fraction of p/q in the following manner:

$$(21) \quad M_k(x, y) = m_k x + \ell_k y = \begin{cases} qy, & k = -1 \\ (-1)^k (q_k x + (pq_k - qp_k)y), & 0 \leq k \leq n-1 \\ (-1)^n qx, & k = n. \end{cases}$$

Then each part of (20) is computed as follows:

•

$$\begin{aligned} & xy \sum_{k=-1}^{n-1} (-1)^{k+1} \sum_{i=0}^{N-2} \frac{B_{i+1} B_{N-i-1}}{(i+1)!(N-i-1)!} (m_k x + \ell_k y)^{N-2-i} (m_{k+1} x + \ell_{k+1} y)^i \\ &= \sum_{k=-1}^{n-1} (-1)^{k+1} \sum_{\alpha, \beta} f_k(\alpha, \beta) x^{\alpha+\beta+1} y^{N-1-\alpha-\beta} \end{aligned}$$

where

$$f_k(\alpha, \beta) := \frac{m_k^\beta m_{k+1}^\alpha}{\alpha! \beta!} \sum_{i=\alpha}^{N-2-\beta} \frac{1}{(N-2-i-\beta)!} \frac{1}{(i-\alpha)!} \frac{B_{i+1}}{(i+1)!} \frac{B_{N-i-1}}{(N-i-1)!} \ell_k^{N-2-i-\beta} \ell_{k+1}^{i-\alpha}$$

•

$$\begin{aligned} & xy \left(\sum_{k=0}^{n-1} (-1)^k a_{k+1} \sum_{i=0}^{N-2} (m_{k-1} x + \ell_{k-1} y)^{N-i-2} (m_{k+1} x + \ell_{k+1} y)^i \right) \\ &= \sum_{k=0}^{n-1} (-1)^k a_{k+1} \sum_{\alpha, \beta} g_k(\alpha, \beta) x^{\alpha+\beta+1} y^{N-1-\alpha-\beta}, \end{aligned}$$

where

$$g_k(\alpha, \beta) := \frac{m_{k-1}^\beta m_{k+1}^\alpha}{\alpha! \beta!} \sum_{i=\alpha}^{N-2-\beta} \frac{(N-i-2)!}{(N-2-i-\beta)!} \frac{i!}{(i-\alpha)!} \ell_{k-1}^{N-i-2-\beta} \ell_{k+1}^{i-\alpha}.$$

•
•

$$M_0^{N-1}x = \sum_{i=0}^{N-1} \binom{N-1}{i} m_0^i \ell_0^j x^{i+1} y^j.$$

$$M_{n-1}^{N-1}y = \sum_{i+j=N-1} \binom{N-1}{i} m_{n-1}^i \ell_{n-1}^j x^i y^{j+1}$$

Theorem 5.1. *The coefficient of $x^{e+1}y^{f+1}$ of Todd_{pq}^N with $e+f=N-2$ is*

$$\begin{aligned} q \sum_{\alpha+\beta=e} \left[\sum_{k=-1}^{n-1} (-1)^{k+1} f_k(\alpha, \beta) + \frac{B_N}{N!} \sum_{k=0}^{n-1} (-1)^k a_{k+1} g_k(\alpha, \beta) \right] \\ + \frac{B_N}{N!} \binom{N-1}{f+1} p^{f+1} + \frac{B_N}{N!} \binom{N-1}{e+1} ((-1)^{n-1} q_{n-1})^{e+1}. \end{aligned}$$

Combining the formula (10) and Thm. 5.1, we have the proof of Thm.1.1.

6. APPLICATION 1: PARTIAL ZETA VALUES

In this section, we apply the formula for generalized Dedekind sums to the formula for partial zeta values of ideals of real quadratic fields at nonpositive integers due to Siegel and finish the proof of Thm.1.2.

Let us recall the definition of the partial zeta function of an ideal of a number field. For an ideal \mathfrak{a} of a number field K , the partial zeta function $\zeta(\mathfrak{a}, s)$ is defined as

$$\zeta(\mathfrak{a}, s) := \sum_{\substack{\mathfrak{b} \sim \mathfrak{a} \\ \mathfrak{b}: \text{integral ideal}}} N\mathfrak{b}^{-s}, \quad \text{for } \text{Re}(s) > 1.$$

$\zeta(\mathfrak{a}, s)$ is meromorphically continued to the whole complex plane with a unique simple pole at $s = 1$.

Let $\mathfrak{a} = [\alpha, \beta]$ be an ideal of a real quadratic field K . Let $\omega = -\frac{\beta}{\alpha} > \omega'$ and $\epsilon > 1$ be the totally positive fundamental unit. With respect to the basis α, β , the multiplication by ϵ^{-1} is represented by a hyperbolic element $\begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$:

$$\epsilon^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

In this setting, a reformulation of the formula of Siegel by Coates-Sinnott (Lemma 11 in [6]) is given as follows:

Theorem 6.1 (Siegel's formula).

$$(22) \quad (-1)^N \frac{2N}{B_{2N}} \zeta(\mathfrak{a}, 1-N) = \frac{1}{q^{2N-1}} \sum_{k=0}^{2N-2} (-1)^k \frac{e_{N,k}}{k+1} (p+s)^{k+1} + \frac{B_{2N}}{2N} \sum_{k=0}^{2N-2} (-1)^{k+1} e_{N,k} \frac{s_{k+1, 2N-k-1}(p, q)}{(k+1)(2N-1-k)}$$

where $e_{N,k}$ is given in the following way:

$$(z^2 + (p+s)z + 1)^{N-1} = \sum_{i=0}^{2N-2} e_{N,k} z^k$$

Now we apply our formula in Thm.1.1 to Siegel's formula. Soon we will see that the contribution of the rational part $s_{e,f}^R(p, q)$ of Dedekind sums canceled by themselves. It is essentially due to the following observation.

Lemma 6.2.

$$\sum_{k=0}^{2N-2} (-1)^k \frac{e_{N,k}}{k+1} (p+s)^{k+1} + \frac{2N}{B_{2N}} \sum_{k=0}^{2N-2} (-1)^{k+1} \frac{e_{N,k}}{(k+1)(2N-1-k)} s_{k+1,2N-k-1}^R(p,q) = 0$$

Proof. We note that from Theorem 1.1, we have

$$s_{e,f}^R(p,q) = \frac{B_{e+f}}{(e+f)} \left(ep^f + f \left((-1)^{n-1} q_{n-1} \right)^e \right).$$

Also we have $s = (-1)^{n-1} q_{n-1}$ (See pp. 39–40, [22]).

Putting these into the left hand side of the equality, we have

$$\begin{aligned} T &:= \sum_{k=0}^{2N-2} (-1)^k \frac{e_{N,k}}{k+1} (p+s)^{k+1} + \frac{2N}{B_{2N}} \sum_{k=0}^{2N-2} (-1)^{k+1} \frac{e_{N,k}}{(k+1)(2N-1-k)} s_{k+1,2N-k-1}^R(p,q) \\ &= \sum_{k=0}^{2N-2} (-1)^k \frac{e_{N,k}}{k+1} (p+s)^{k+1} + \sum_{k=0}^{2N-2} (-1)^{k+1} \frac{e_{N,k}}{(k+1)(2N-1-k)} ((k+1)p^{2N-k-1} + (2N-k-1)s^{k+1}). \end{aligned}$$

We notice that

$$\int_0^Z (z^2 + (p+s)z + 1)^{N-1} dz = \sum_{i=0}^{2N-2} \frac{e_{N,k}}{k+1} Z^{k+1}$$

and

$$e_{N,2N-2-j} = e_j, \quad j = 0, 1, \dots, 2N-2.$$

Now T can be written as sum of definite integrals as follows:

$$\begin{aligned} T &= - \sum_{k=0}^{2N-2} (-1)^{k+1} \frac{e_{N,k}}{k+1} (p+s)^{k+1} + \sum_{k=0}^{2N-2} (-1)^{k+1} \frac{e_{N,k}}{(k+1)(2N-1-k)} ((k+1)p^{2N-k-1} + (2N-k-1)s^{k+1}) \\ &= - \int_0^{-p-s} (z^2 + (p+s)z + 1)^{N-1} dz + \int_0^{-p} (z^2 + (p+s)z + 1)^{N-1} dz + \int_0^{-s} (z^2 + (p+s)z + 1)^{N-1} dz \\ &= - \int_{-p}^{-p-s} (z^2 + (p+s)z + 1)^{N-1} dz + \int_0^{-s} (z^2 + (p+s)z + 1)^{N-1} dz \\ &= - \int_0^{-s} (z^2 + (-p+s)z + 1 - ps)^N - (z^2 + (p+s)z + 1)^N dz. \end{aligned}$$

Finally it remains to show that

$$\int_0^{-s} (z^2 + (-p+s)z + 1 - ps)^N - (z^2 + (p+s)z + 1)^N dz = 0.$$

By change of the variable $z = t - s/2$, the above becomes

$$\int_{s/2}^{-s/2} \left((t^2 - pt + A)^N - (t^2 + pt + A)^N \right) dt$$

where $A = 1 - ps/2 - s^2/4$. Further the integrand is simplified as

$$-2pt \sum_{i+j=N-1} (t^2 - pt + A)^i (t^2 + pt + A)^j.$$

It is easily checked to be an odd function in t . Thus the vanishing of T is obtained and this completes the proof. \square

Now simply applying the above Lemma to Siegel's formula, we finish the proof of Thm.1.2 in the following:

$$\begin{aligned}
& \frac{(-1)^N 2N}{B_{2N}} \zeta(a, 1-N) \\
&= \frac{-1}{q^{2N-1}} \sum_{k=0}^{2N-2} (-1)^{k+1} \frac{e_{N,k}}{k+1} (p+s)^{k+1} \\
&\quad + \frac{1}{q^{2N-1}} \frac{2N}{B_{2N}} \sum_{k=0}^{2N-2} (-1)^{k+1} \frac{e_{N,k}}{(k+1)(2N-1-k)} \left(q s_{k+1,2N-k-1}^I(p,q) + s_{k+1,2N-k-1}^R(p,q) \right) \\
&= \frac{-1}{q^{2N-1}} \sum_{k=0}^{2N-2} (-1)^{k+1} \frac{e_{N,k}}{k+1} (p+s)^{k+1} + \frac{1}{q^{2N-1}} \frac{2N}{B_{2N}} \sum_{k=0}^{2N-2} (-1)^{k+1} \frac{e_{N,k}}{(k+1)(2N-1-k)} s_{k+1,2N-k-1}^R(p,q) \\
&\quad + \frac{(-1)^N 2N}{q^{2N-2} B_{2N}} \sum_{k=0}^{2N-2} (-1)^{k+1} e_{N,k} \frac{s_{k+1,2N-k-1}^I(p,q)}{(k+1)(2N-1-k)} \\
&= \frac{(-1)^N 2N}{q^{2N-2} B_{2N}} \sum_{k=0}^{2N-2} (-1)^{k+1} e_{N,k} \frac{s_{k+1,2N-k-1}^I(p,q)}{(k+1)(2N-1-k)}
\end{aligned}$$

Next comes explicit formula of the partial zeta values for $s = -1$ and $s = -2$ using integral part of Dedekind sum and obtain explicit expression for them using continued fraction of $\frac{p}{q}$.

Example 6.3.

$$\begin{aligned}
\zeta(a, -1) &= -\frac{1}{3q^3} s_{1,3}^I(p,q) + \frac{(p+s)}{4q^3} s_{2,2}^I(p,q) - \frac{1}{3q^3} s_{3,1}^I(p,q) \\
\zeta(a, -2) &= \frac{1}{5q^5} s_{1,5}^I(p,q) - \frac{1}{4q^5} (p+s) s_{2,4}^I(p,q) + \frac{1}{9q^5} (p+s)^2 s_{3,3}^I(p,q) - \frac{1}{4q^5} (p+s) s_{4,2}^I(p,q) + \frac{1}{5q^5} s_{5,1}^I(p,q).
\end{aligned}$$

Below is the table of explicit formula of $s_{i,j}^I(p,q)$ for $i+j=4$ and 6 , respectively.

$s_{2,2}^I(p,q)$	$\frac{1}{36} \sum_{k=-1}^{n-1} (-1)^{k+1} \left(D_{k+1} q_k + D_k q_{k+1} \right) - \frac{1}{180} \sum_{k=0}^{n-1} (-1)^k a_{k+1} \left(q_{k+1} (D_{k-1} + 2D_{k+1}) + q_{k-1} (2D_{k-1} + D_{k+1}) \right)$
$s_{3,1}^I(p,q)$	$\frac{1}{24} \sum_{k=-1}^{n-1} (-1)^k q_k q_{k+1} - \frac{1}{120} \sum_{k=0}^{n-1} (-1)^k a_{k+1} \left(q_{k+1}^2 + q_{k-1} q_{k+1} + q_{k-1}^2 \right)$
$s_{1,3}^I(p,q)$	$\frac{1}{24} \sum_{k=-1}^{n-1} (-1)^k D_k D_{k+1} - \frac{1}{120} \sum_{k=0}^{n-1} (-1)^k a_{k+1} \left(D_{k+1}^2 + D_{k-1} D_{k+1} + D_{k-1}^2 \right)$

$s_{5,1}^I(p, q)$	$\frac{1}{72} \sum_{k=-1}^{n-1} (-1)^{k+1} (q_{k+1} q_k^3 + q_k q_{k+1}^3)$ $+ \frac{1}{252} \sum_{k=0}^{n-1} (-1)^k a_{k+1} (q_{k-1}^4 + q_{k-1}^3 q_{k+1} + q_{k-1}^2 q_{k+1}^2 + q_{k-1} q_{k+1}^3 + q_{k+1}^4)$
$s_{4,2}^I(p, q)$	$\frac{1}{180} \sum_{k=-1}^{n-1} (-1)^{k+1} (D_{1+k} q_k^3 + 3D_k q_k^2 q_{1+k} + 3D_{1+k} q_k q_{1+k}^2 + D_k q_{1+k}^3)$ $+ \frac{1}{630} \sum_{k=0}^{n-1} (-1)^k a_{k+1} (4D_{k-1} q_{k-1}^3 + D_{k+1} q_{k-1}^3 + 3D_{k-1} q_{k-1}^2 q_{k+1} + 2D_{k+1} q_{k-1}^2 q_{k+1}$ $+ 2D_{k-1} q_{k-1} q_{k+1}^2 + 3D_{k+1} q_{k-1} q_{k+1}^2 + D_{k-1} q_{k+1}^3 + 4D_{k+1} q_{k+1}^3)$
$s_{3,3}^I(p, q)$	$\frac{1}{80} \sum_{k=-1}^{n-1} (-1)^{k+1} (D_k D_{1+k} q_k^2 + D_k^2 q_k q_{k+1} + D_{k+1}^2 q_k q_{1+k} + D_k D_{1+k} q_{1+k}^2) +$ $\frac{1}{840} \sum_{k=0}^{n-1} (-1)^k a_{k+1} (6D_{k-1}^2 q_{k-1}^2 + 3D_{k-1} D_{k+1} q_{k-1}^2 + D_{k+1}^2 q_{k-1}^2 + 3D_{k-1}^2 q_{k-1} q_{1+k}$ $+ 4D_{k-1} D_{k+1} q_{k-1} q_{k+1} + 3D_{k+1}^2 q_{k-1} q_{k+1} + D_{k-1}^2 q_{k+1}^2 + 3D_{k-1} D_{k+1} q_{k+1}^2 + 6D_{1+k}^2 q_{1+k}^2)$
$s_{2,4}^I(p, q)$	$\frac{1}{180} \sum_{k=-1}^{n-1} (-1)^{k+1} (3D_k^2 D_{k+1} q_k + D_{k+1}^3 q_k + D_k^3 q_{1+k} + 3D_k D_{1+k}^2 q_{1+k})$ $+ \frac{1}{630} \sum_{k=0}^{n-1} (-1)^k a_{k+1} (4D_{k-1}^3 q_{k-1} + 3D_{k-1}^2 D_{k+1} q_{k-1} + 2D_{k-1} D_{k+1}^2 q_{k-1} + D_{k+1}^3 q_{k-1}$ $+ D_{k-1}^3 q_{k+1} + 2D_{k-1}^2 D_{k+1} q_{k+1} + 3D_{k-1} D_{k+1}^2 q_{k+1} + 4D_{k+1}^3 q_{k+1})$
$s_{1,5}^I(p, q)$	$\frac{q}{72} \sum_{k=-1}^{n-1} (-1)^{k+1} (D_{k+1} D_k^3 + D_k D_{k+1}^3)$ $+ \frac{q}{252} \sum_{k=0}^{n-1} (-1)^k a_{k+1} (D_{k-1}^4 + D_{k-1}^3 D_{k+1} + D_{k-1}^2 D_{k+1}^2 + D_{k-1} D_{k+1}^3 + D_{k+1}^4)$

7. APPLICATION 2: EQUIDISTRIBUTION OF THE GRAPH OF GENERALIZED DEDEKIND SUM

In this section, we give a proof for Thm.1.3 which describes the distribution of $H_{ij}(p, q)$ for $i + j$ even and a relatively prime pair of integers (p, q) in \mathbb{R} given as follows:

$$H_{ij}(p, q) := \left(\frac{p}{q}, R_{i+j} q^{N-2} s_{i,j}(p, q) \right)$$

We will show that the fractional part of the graph $H_{ij}(p, q)$ taken in $[0, 1]^2$ are equidistributed in the sense of Weyl([24]). As a function on the unit torus, it is equivalent to saying that the average of the point-mass supported on the sequence of the vectors $H_{ij}(p, q)$ (ordered by increment of q) is weakly convergent to the standard Lebesgue measure on the torus.

A sequence α_k in $[0, 1)$ is said to be equidistributed if the average of the point-mass supported on the truncated sequence weakly converges to the standard Lebesgue measure of the unit interval. It is checked by the behavior of the Fourier coefficient of the average of the point-mass. Let $\mathbf{e}(x) = \exp(2\pi i x)$ for $x \in \mathbb{R}$. The Fourier coefficient of the character $\mathbf{e}(mx)$ is given as

$$\int_0^1 \mathbf{e}(mx) \frac{1}{N} \sum_{k \leq N} \delta_{\alpha_k} dx = \frac{1}{N} \sum_{k \leq N} \mathbf{e}(m\alpha_k).$$

Then the sequence α_k is equidistributed iff $a_m(N) \rightarrow 0$ as $N \rightarrow \infty$ for each nonzero $m \in \mathbb{Z}$.

For a sequence on a n -dimensional torus to be equidistributed, the criterion is changed as follows. We note first that a character of n -dimensional is represented by a lattice vector $\mathbf{m} \in \mathbb{Z}^n$. It needs that for any $\mathbf{m} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$,

$$\frac{1}{N} \sum_{k \leq N} \mathbf{e}(\mathbf{m} \cdot \alpha_k) \rightarrow \mathbf{0} \quad \text{as } N \rightarrow \infty.$$

Now we return to the case of our interest. Let $\mathbf{m} \in \mathbb{Z}^2$ a lattice vector and let, for a positive real number x , $E(\mathbf{m}, x)$ be the average of the exponential of $(2\pi i) \mathbf{m} \cdot H_{ij}(p, q)$:

$$(23) \quad E(\mathbf{m}, x) := \frac{1}{\#\{(p, q) \mid \gcd(p, q) = 1, p < q \leq x\}} \sum_{0 < q < x} \sum_{\substack{0 < p < q \\ (p, q) = 1}} \mathbf{e}(\mathbf{m} \cdot H_{ij}(p, q)).$$

For $H_{ij}(p, q)$ to be equidistributed in T^2 , it needs to check that $E(\mathbf{m}, x) \rightarrow 0$ as $x \rightarrow \infty$ for every nonzero \mathbf{m} . This will be shown by relating an exponential sum.

Now, we relate

$$(24) \quad \sum_{\substack{0 < p < q \\ (p, q) = 1}} \mathbf{e}(\mathbf{m} \cdot H_{ij}(p, q))$$

to an exponential sum for a Laurent polynomial.

For a Laurent polynomial $F(x) \in \mathbb{Z}[x, x^{-1}]$, we denote its exponential sum over $\mathbb{Z}/q\mathbb{Z}$ by $K(F, q)$:

$$(25) \quad K(F, q) := \sum_{x \in (\mathbb{Z}/q\mathbb{Z})^*} \mathbf{e}_q(F(x)),$$

where $\mathbf{e}_q(x) := \mathbf{e}(\frac{x}{q}) = \exp(2\pi i \frac{x}{q})$.

For $H_{ij}(p, q)$, we recall the formula for the fractional part of $s_{ij}(p, q)$.

Theorem 7.1 (Jun-Lee[14]). *For positive integers i, j with $i + j = N \geq 2$ even, we have*

$$R_N q^{N-2} s_{ij}(p, q) - \alpha_N r_N \frac{p^{i(N-1)} + p^{j(N-1)}}{q} \in \mathbb{Z},$$

where p' is an integer such that $p'p \equiv 1 \pmod{q}$.

Thus one can relate the exponential sum below to $\mathbf{m} \cdot H_{ij}(p, q)$.

$$\sum_{\substack{0 < p < q \\ (p, q) = 1}} \mathbf{e}(\mathbf{m} \cdot H_{ij}(p, q)) = K(\mathbf{m} \cdot F_{ij}, q).$$

where $F_{ij}(x)$ is the rank-2 vector of Laurent polynomials

$$F_{ij}(x) = \left(x, x^{-i} \alpha_N r_N \binom{N-1}{i} + x^j \alpha_N r_N \binom{N-1}{j} \right),$$

and for $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2 \setminus \{0\}$,

$$(26) \quad \mathbf{m} \cdot F_{ij}(x) = \sum_{i=1}^{N-1} m_1 x + m_2 x^{-i} \alpha_N r_N \binom{N-1}{i} + m_2 x^j \alpha_N r_N \binom{N-1}{j}.$$

To estimate the exponential sum $K(\mathbf{m} \cdot F_{ij}, q)$, we recall a result from SGA 4 $^{1/2}$ ([8]). Consider an exponential sum of a Laurent polynomial $f(x)$ over $\mathbb{P}^1(\mathbb{F}_p)$ for a prime p :

$$S'_f := \sum_{\substack{x \in \mathbb{P}^1(\mathbb{F}_p) \\ f(x) \neq \infty}} \mathbf{e}_p(f(x))$$

The irregularity index $v_x^*(f)$ at x is actually the Swan conductor of the associated ℓ -adic sheaf to f . It is a local invariant defined in the following way:

- (a) $v_x(f) = \begin{cases} \text{the order of pole at } x \text{ of } f & \text{if } f(x) = \infty \\ 0, & \text{otherwise} \end{cases}$
- (b) $v_x^*(f) := \inf_g v_x(f + g^p - g)$

A modified exponential sum is defined from S'_f as

$$S_f := \sum_{\substack{x \in \mathbb{P}^1(\mathbb{F}_p) \\ v_x^*(f) = 0}} \mathbf{e}_p(f(x))$$

where the value of $\mathbf{e}_p(f(x))$ is extended to $\{x | v_x^*(f) = 0\}$ by putting

$$\mathbf{e}_p(f(x)) := \mathbf{e}_p((f + g^p - g)(x))$$

for g which makes $f + g^p - g$ regular at x .

Applying the formula (3.5.2) of *loc.cit.*, we obtain a Weil type bound of S_f :

$$(27) \quad |S_f| \leq -2 + \sum_{v_x^*(f) \neq 0} (1 + v_x^*(f)) p^{1/2}$$

In our case, $f(x) = \mathbf{m} \cdot F_{ij}(x)$ explicitly given in (26).

$$v_0^*(f) = \begin{cases} 0 & \text{if } m_2 = 0 \\ i & \text{Otherwise} \end{cases} \quad \text{and} \quad v_\infty^*(f) = \begin{cases} j & \text{if } j > 1 \text{ and } m_2 \neq 0 \\ 1 & \text{if } m_2 = 0 \\ 0 & \text{if } j = 1 \text{ and } m_1 + m_2 = 0 \end{cases}$$

For any nonzero \mathbf{m} , we have an upper bound for the exponential sum:

$$(28) \quad |K(\mathbf{m} \cdot F_{ij}, p)| \leq C \cdot p^{1/2}$$

for a given prime p and C is a constant independent of p .

Now, we consider the case that q is a power of prime p or product of several prime powers. Using the same reduction process in Section 6 of [14], we obtain the following bound for given positive integer q .

Theorem 7.2. *For given positive integer i, j such that $N = i + j$ is even and positive integer q , we have*

$$|K(\mathbf{m} \cdot F_{ij}, q)| < q^{\frac{3}{4} + \epsilon}$$

for all $\epsilon > 0$.

Finally the Weyl criterion for $H_{ij}(p, q)$ comes from the following estimate

$$(29) \quad \sum_{0 < q < x} \sum_{\substack{0 < p < q \\ (p, q) = 1}} \mathbf{e}(\mathbf{m} \cdot H_{ij}(p, q)) = \sum_{0 < q < x} K(\mathbf{m} \cdot F_{ij}, q) \leq x^{\frac{7}{4} + \epsilon}.$$

For $x > 1$ let $\phi(x) := |(\mathbb{Z}/[x]\mathbb{Z})^*|$ be the Euler's phi function. Since $\sum_{q < x} \phi(q) \sim x^2$, the criterion is fulfilled for the fractional part of the vector $H_N(p, q)$

$$E(m, x) = \frac{1}{\#\{(p, q) | (p, q) = 1, 0 < p < q \leq x\}} \sum_{0 < q < x} \sum_{\substack{0 < p < q \\ (p, q) = 1}} \mathbf{e}(m \cdot R_N H_{ij}(p, q)) \rightarrow 0,$$

as $x \rightarrow \infty$. Therefore the proof is done.

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